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Dynamical Symmetries of the Geodesic Equation

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A class of dynamical symmetries for the Euler–Lagrange equations corresponding to the Lagrangian $L = (1/2)g_{ab}\dot{q}^a\dot{q}^b$ is determined. The members of the class are closely related to tensor fields defined on the configuration space. First integrals generated by the dynamical symmetries through deformation of a given first integral are then examined. Noether-type conserved quantities whose expression depends only on the dynamical symmetry are also explicitly exhibited. Applications to general relativity are also pointed out in the course of the discussion.

1. INTRODUCTION

In the last few years there has been considerable progress in the development of procedures leading to the generation of conserved quantities for classical Lagrangian systems, with many new ideas being introduced. A particular emphasis has been given to the problem of relating constants of the motion to the various types of symmetry that may be considered (Sarlet et al., 1981). However, in spite of the interesting theoretical results already achieved, there seems to be a lack of practical procedures that provide a way of constructing symmetry generators and of determining specific conserved quantities (Leach, 1981).

The aim of this paper is to give a contribution toward filling these gaps. In so doing, besides exploring new applications of the existing theory, we will also obtain additional insights into the efficiency of already established theoretical results, which in turn will suggest new lines of development for the theoretical analysis.

The starting point of our approach is the problem of determining dynamical symmetries for the Euler–Lagrange equations generated by the

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regular Lagrangian $L = \frac{1}{2} g_{ab} \dot{q}^a \dot{q}^b$. The form of the Lagrangian is sufficiently general to encompass a certain number of relevant physical applications. For instance, the Lagrangian L is known to model the geodesic motion of freely falling particles in the general theory of relativity; in analytical mechanics it governs the free motion of a mechanical system with a finite number of degrees of freedom; moreover, it may be used to describe the motion of a conservative Newtonian system, modulo a suitable conformal transformation in the configuration space (Xanthopoulos, 1982). Even though in the course of this paper we shall mainly adopt the terminology of general relativity, it is to be noticed that the results do not depend on the signature of the metric, unless explicitly stated otherwise.

The literature on the relations between symmetry properties of the geodesic equation and conserved quantities is very extended, in view of the fact that the determination of the geodesics of a given metric is a fundamental problem in the field of general relativity (see, e.g., the list of references in Katzin et al., 1981; and in Caviglia et al., 1982a). In general, however, this problem has been dealt with using very special ad hoc methods, and no systematic treatment seems as yet to be available. It turns out that by the introduction of techniques coming from modern analytical mechanics we obtain an approach that unifies already existing results, and we also find new families of first integrals of motion.

From the point of view of analytical mechanics, it is to be noticed that we will not determine the complete set of dynamical symmetries for the given set of Euler–Lagrange equations. More precisely, we shall look for those dynamical symmetries that are closely related to tensor fields defined on the configuration manifold, following suggestions coming from general relativity. Of course, this procedure may be extended to more general cases. We want also to point out that the dynamical symmetries of the above class turn out to be connected in a very natural way to the so-called "Noether-type" conserved quantities (Lutzky, 1979), which may be written down explicitly in terms of the given dynamical symmetry. More information on this point will be given in Section 7.

2. PRELIMINARIES ON LAGRANGIAN SYSTEMS AND DYNAMICAL SYMMETRIES

This section is devoted to a brief review of some basic results concerning Lagrangian systems. A more exhaustive treatment may be found in works by Crampin (1977) and by Sarlet and Cantrijn (1981).

Let M be an n-dimensional manifold and let $R \times TM$ be the associated extended tangent bundle, referred to local natural coordinates $(s, q^a, \dot{q}^a)$

($a = 1, \ldots, n$). A regular Lagrangian L is a function defined on $R \times TM$ such that the matrix $\partial^2 L/(\partial \dot{q}^a \partial \dot{q}^b)$ is nonsingular in the domain of the given local coordinates. The Euler–Lagrange equations corresponding to a regular Lagrangian L,

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{q}^a} - \frac{\partial L}{\partial q^a} = 0, \qquad a = 1, \ldots, n \tag{1}$$

may be written in the equivalent normal form

$$\frac{d\dot{q}^a}{ds} = \Lambda^a(s, q, \dot{q}), \qquad a = 1, \ldots, n \tag{2}$$

where Λ^a is defined by

$$\Lambda^a(s, q, \dot{q}) = g^{ab}\left(-\frac{\partial^2 L}{\partial \dot{q}^b \partial q^c}\dot{q}^c - \frac{\partial^2 L}{\partial \dot{q}^b \partial s} + \frac{\partial L}{\partial q^b}\right) \tag{3}$$

and g^{ab} is the inverse matrix of $\partial^2 L/(\partial \dot{q}^a \partial \dot{q}^b)$. (In the following sections the entries g^{ab} will coincide with the contravariant components of the metric tensor, as suggested by the notation introduced.)

The motions of the mechanical system described by the Lagrangian L are the projections onto M of the integral curves of the vector field Γ of local expression

$$\Gamma = \frac{\partial}{\partial s} + \dot{q}^a \frac{\partial}{\partial q^a} + \Lambda^a \frac{\partial}{\partial \dot{q}^a} \tag{4}$$

In general, the knowledge of the symmetries of the system of differential equations defined by equation (4) is of great help in the process of determining the integral curves of Γ. Actually, besides giving information about peculiar properties of the system of equations that is being analyzed, symmetries may be used either to find special solutions or to reduce the system to a simpler one (Hermann, 1968; Muncaster, 1982).

A vector field Y on $R \times TM$ with local representation

$$Y = \tau(s, q, \dot{q})\frac{\partial}{\partial s} + K^a(s, q, \dot{q})\frac{\partial}{\partial q^a} + \eta^a(s, q, \dot{q})\frac{\partial}{\partial \dot{q}^a} \tag{5}$$

is said to be a *dynamical symmetry* of Γ if and only if

$$\mathfrak{L}_Y \Gamma = [Y, \Gamma] = g\Gamma \tag{6}$$

for some suitable function g.

By substitution of equations (4) and (5) into (6) it may be shown that equation (6) is equivalent to the conditions

$$\eta^a = \Gamma(K^a) - \dot{q}^a \Gamma(\tau) \tag{7}$$

$$\Gamma(\eta^a) - \Lambda^a \Gamma(\tau) - Y(\Lambda^a) = 0 \tag{8}$$

$$g = -\Gamma(\tau) \tag{9}$$

The dynamical symmetry Y is said to be a *point symmetry* of Γ if and only if the components τ and K^a depend only on q and s (Lutzky, 1978, 1982; Gonzalez Gascon, 1980).

Finally, let us consider the 1-form θ defined by

$$\theta = L\,ds + \frac{\partial L}{\partial \dot{q}^a}(dq^a - \dot{q}^a\,ds) \tag{10}$$

A *Noether symmetry* is generated by a vector field Y of the form (5) satisfying the condition

$$\mathfrak{L}_Y\, d\theta = 0 \tag{11}$$

It may be shown that every Noether symmetry is also a dynamical symmetry (Sarlet et al., 1981), but the converse is not true. In general, both Noether symmetries and point symmetries identify first integrals of motion (Lutzky, 1979b, 1979c, 1982; Sarlet et al., 1981; Crampin, 1977). On the contrary, the relations between dynamical symmetries and conserved quantities are much more involved (Lutzky, 1979a; Sarlet et al., 1981). In the following sections we shall introduce conserved quantities of geodesic motion concomitant with dynamical symmetries.

3. A CLASS OF DYNAMICAL SYMMETRIES FOR THE GEODESIC EQUATION

It is well known that, in general relativity, the motion of freely falling particles is modeled by the Euler–Lagrange equations deduced from a regular Lagrangian L of the form $L = \frac{1}{2} g_{ab}(q)\dot{q}^a\dot{q}^b$, where g_{ab} is a Lorentzian metric. The equations of motion may be written as

$$\frac{d\dot{q}^a}{ds} = -\Gamma_{bc}{}^a \dot{q}^b \dot{q}^c = \Lambda^a(q, \dot{q}) \tag{12}$$

where the symbols $\Gamma_{bc}{}^a$ denote, as usual, the connection coefficients. Of course, we obtain the geodesics of the metric g_{ab} by projecting onto M the integral curves of Γ, with Λ^a given by equation (12).

During the past few years there has been a considerable amount of progress in the study of those vector fields of M that, in some sense, may be regarded as symmetry generators for the geodesic equation (12), with many new ideas and ingenious methods for the determination of constants of motion being introduced. At this stage, however, it seems more convenient to formulate the problem in more abstract terms, taking profit of recent developments in the field of analytical mechanics. Accordingly, we determine a family of dynamical symmetries Y of the field (4), with Λ^a given by equation (12). It will be shown that the family is sufficiently extended to contain and unify earlier results already obtained in the field of general relativity. Moreover, the members of the family may be used to construct new classes of previously unknown constants of geodesic motion. Finally, we shall also obtain additional insights into the connections between dynamical symmetries and first integrals of motion.

Let K^a be a vector field defined on M. Our aim is to find a dynamical symmetry Y such that its natural projection onto M coincides with K^a. If such a Y exists, then equation (7) yields the expression of η^a, namely,

$$\eta^a = \dot{q}^b(\partial K^a/\partial q^b) - \dot{q}^a\Gamma(\tau) \tag{13}$$

By requiring that also equation (8) be satisfied, we may determine τ while, at the same time, we find conditions on K^a ensuring the existence of the dynamical symmetry. Consequently, let us substitute into equation (8) the expressions (13) and (12) of η^a and Λ^a, respectively. After long and rather involved calculations equation (8) can be rewritten in the equivalent form

$$-\dot{q}^a\Gamma\Gamma(\tau) + \left(\frac{d^2K^a}{ds^2} + 2\Gamma_{bc}{}^a\frac{dK^c}{ds}\dot{q}^b + \frac{\partial\Gamma_{bc}{}^a}{\partial q^d}K^d\dot{q}^b\dot{q}^c\right) = 0 \tag{14}$$

Recalling the expression of the Riemann tensor $R^a{}_{bcd}$ in terms of the connection coefficients, equation (14) can be cast into the simpler form (Shirokov, 1973)

$$\dot{q}^c\dot{q}^b\nabla_c\nabla_b K^a + R^a{}_{bcd}\dot{q}^b K^c\dot{q}^d = \dot{q}^a\Gamma\Gamma(\tau) \tag{15}$$

where we recall that the quantities $\dot{q}^b$ are tangent to a geodesic of M, since equation (12) is supposed to hold.

The left-hand side of equation (15) has exactly the form of the equation of geodesic deviation. Accordingly, in the case of vanishing $\Gamma\Gamma(\tau)$, the restriction of K^a to an arbitrary geodesic of M must be a Jacobi field. Under generic conditions, it has been shown that a projective collineation (PC) is the most general vector field such that the expression in the left-hand side of (15) is proportional to the tangent vector of the fiducial geodesic, whatever be the choice of the geodesic (Caviglia et al., 1982a). Recalling that a PC is characterized by the condition

$$\nabla_c(\nabla_a K_b + \nabla_b K_a) = (2g_{ab}\nabla_c\nabla_d K^d + g_{ac}\nabla_b\nabla_d K^d + g_{bc}\nabla_a\nabla_d K^d)/(n+1) \tag{16}$$

equation (15) may be rewritten as

$$(\delta^a_b \nabla_c\nabla_d K^d + \delta^a_c \nabla_b\nabla_d K^d)\dot{q}^b\dot{q}^c/(n+1) = \dot{q}^a\Gamma\Gamma(\tau) \tag{17}$$

from which it follows that we may put

$$\Gamma(\tau) = 2\nabla_d K^d/(n+1) \tag{18}$$

Since equation (18) may always be solved for τ, substituting into equation (13) and comparing with equation (5) we obtain the expression of the dynamical symmetry associated with K^a. To summarize, we have found a new characterization of PCs as generators of dynamical symmetries.

Suppose now that we are looking for point symmetries: according to the definition, τ is not allowed to depend on $\dot{q}$. Then, on account of equation (4), equation (18) yields

$$\frac{\partial\tau}{\partial s} + \dot{q}^a\frac{\partial\tau}{\partial q^a} = \frac{2\nabla_d K^d}{n+1} \tag{19}$$

In view of the fact that the right-hand side of equation (19) is independent of $\dot{q}$, it follows that $\tau = \tau(s)$. Substituting again into equation (19) we obtain the compatibility condition

$$\frac{d\tau}{ds} = \text{const} = \frac{2\nabla_d K^d}{n+1} \tag{20}$$

Since equation (20) shows that the PC K^a reduces to an affine collineation (AC) (Katzin et al., 1972), it follows that point symmetries are generated by ACs.

To conclude this section we shall examine two particular cases. Firstly, let us assume that the AC K^a degenerates into a Killing vector: this means that the condition

$$\nabla_a K_b + \nabla_b K_a = 0 \tag{21}$$

holds identically, so that it follows from equation (18) that $\Gamma(\tau) = 0$. Accordingly, the Killing vector K^a identifies the point symmetry

$$Y = K^a \frac{\partial}{\partial q^a} + \dot{q}^b \frac{\partial K^a}{\partial q^b} \frac{\partial}{\partial \dot{q}^a} \tag{22}$$

It may be verified by direct calculation that $\mathfrak{L}_Y \theta = 0$. Using the well-known result that $d\mathfrak{L}_Y \theta = \mathfrak{L}_Y d\theta$, we may conclude that the Killing vector K^a identifies a Noether symmetry.

Secondly, let us consider the null vector field on M. In view of the previous discussion it gives rise to the dynamical symmetry

$$Y = \frac{\partial}{\partial s} \tag{23}$$

that will be used in the following sections in order to generate conserved quantities.

4. DYNAMICAL SYMMETRIES CONCOMITANT WITH TENSOR FIELDS OF THE CONFIGURATION SPACE

In this section we look for dynamical symmetries of Γ such that K^a is a polynomial function of $\dot{q}$ given by

$$K^a = K^a{}_{a_2 \cdots a_p} \dot{q}^{a_2} \cdots \dot{q}^{a_p} \tag{24}$$

where $K_{a_1 \cdots a_p}$ is a totally symmetric tensor field defined on M and, of course, indices are raised by using the contravariant metric tensor g^{ab}.

Suppose that a field Y of the form (5) with K^a given by equation (24) is a dynamical symmetry: substituting equation (24) into equation (7) it follows that η^a satisfies the relation

$$\eta^a = \dot{q}^b \frac{\partial}{\partial q^b} \left(K^a{}_{a_2 \cdots a_p} \dot{q}^{a_2} \cdots \dot{q}^{a_p} \right)$$

$$- (p-1) \Gamma_{bc}{}^d K^a{}_{d a_3 \cdots a_p} \dot{q}^b \dot{q}^c \dot{q}^{a_3} \cdots \dot{q}^{a_p} - \dot{q}^a \Gamma(\tau) \tag{25}$$

Using the same procedure of the previous section, let us substitute equations

(24), (25), and (12) into equation (8). After long and cumbersome calculations it may be shown that equation (8) is equivalent to equation (15), where K^a is now given by equation (24). Using recent results of Caviglia et al. (1982a, b) it may be proved that equation (15) is satisfied by a vector K^a of the form (24) along every integral curve of Γ if and only if there exists a totally symmetric tensor field $k_{a_1\cdots a_p}$ such that the following relation holds identically:

$$\left[(p+2)\nabla_{(b}\nabla_a K_{a_1\cdots a_p)} - 2\nabla_a\nabla_{(b}K_{a_1\cdots a_p)}\right]\Big/ p = g_{a(b}k_{a_1\cdots a_p)} \tag{26}$$

and $\Gamma\Gamma(\tau)$ is given by

$$\Gamma\Gamma(\tau) = k_{a_1\cdots a_p}\dot{q}^{a_1}\cdots\dot{q}^{a_p}. \tag{27}$$

Recalling that (Caviglia et al., 1982a) a totally symmetric tensor field $K_{a_1\cdots a_p}$ satisfying equation (26) is usually referred to as a generalized Killing tensor (GKT), we may conclude that every dynamical symmetry Y for which equation (24) holds is related to a GKT. It follows easily that the converse of the last statement also holds, provided one defines K^a and η^a through equations (24) and (25), respectively, while τ is to be determined by solving equation (27).

To complete the discussion, we notice that equation (27) is satisfied by a vanishing function τ if $k_{a_1\cdots a_p} = 0$, i.e., iff

$$\nabla_a\nabla_{(b}K_{a_1\cdots a_p)} = 0 \tag{28}$$

In particular, equation (28) holds for a Killing tensor of order p, which is defined as a totally symmetric tensor of order p satisfying the relation

$$\nabla_{(b}K_{a_1\cdots a_p)} = 0 \tag{29}$$

Killing tensors have long been known as generators of first integrals of geodesic motion; we will show in the following sections how this property also extends to generic GKTs.

5. CONSERVED QUANTITIES ASSOCIATED WITH DYNAMICAL SYMMETRIES BY DEFORMATION PROCESSES

In this section we study the connections between the previously described classes of dynamical symmetries and constants of the motion. We begin by recalling a very simple result, which follows directly from the definition (29) of KT.

Lemma. If $K_{a_1\cdots a_p}$ is a KT then the quantity

$$\phi = K_{a_1\cdots a_p}\dot{q}^{a_1}\cdots\dot{q}^{a_p} \tag{30}$$

is conserved. In particular, the Lagrangian L is a conserved quantity.

A remarkable class of first integrals will now be obtained by a process of deformation of the conserved quantities described in the lemma. To this aim, we notice that to every dynamical symmetry Y and to every conserved quantity ϕ we may associate another first integral given by

$$Y(\phi) = \text{const} \tag{31}$$

along the trajectories of the system. To prove this it suffices to recall that the definition (6) and equation (9) imply

$$Y\Gamma(h) - \Gamma Y(h) = -\Gamma(\tau)\Gamma(h) \tag{32}$$

for every function h. Then equation (31) follows from equation (32) under the assumption that h is a first integral of motion, i.e., $\Gamma(h) = 0$.

As a consequence of the lemma and of equation (31) we shall prove the following corollary.

Corollary. Consider a dynamical symmetry generated by a PC K^a and a Killing tensor $K_{a_1\cdots a_p}$. Then the quantity

$$\left[\pounds_K K_{a_1\cdots a_p} - 2p\nabla_a K^a K_{a_1\cdots a_p}/(n+1)\right]\dot{q}^{a_1}\cdots\dot{q}^{a_p} \tag{33}$$

is a first integral of motion.

Proof. Substitute the expression (30) of ϕ into equation (31) and suppose that Y has the form (5) with η^a given by equation (13). Then we have

$$\begin{aligned} Y(\phi) = \Big[&K^a\left(\partial K_{a_1\cdots a_p}/\partial q^a\right) + p\left(\partial K^b/\partial q^{a_1}\right)K_{ba_2\cdots a_p} \\ &- p\Gamma(\tau)K_{a_1\cdots a_p}\Big]\dot{q}^{a_1}\cdots\dot{q}^{a_p} \end{aligned} \tag{34}$$

Equation (33) follows easily, recalling the definition of the Lie derivative and comparing equation (34) with equation (18). ■

Some comments are in order now. First of all, the conserved quantity (33) does not depend on the explicit determination of the component τ of Y. Then it may be written down immediately, provided one knows the PC and the KT.

Secondly, under the further assumption that the given KT coincides with the metric tensor, the first integral (33) can be written in the equivalent form

$$\tfrac{1}{2}\left[\nabla_a K_b + \nabla_b K_a - 4\nabla_c K^c g_{ab}/(n+1)\right]\dot{q}^a\dot{q}^b \tag{35}$$

showing the existence of a second-order Killing tensor canonically associated to every PC.

Thirdly, the content of the corollary is already known in the literature as the related integral theorem (Katzin et al., 1968, 1973, 1974), which has also been used to derive the conservation of the Runge–Lenz vector in the Kepler problem.

We shall now prove a similar result under the assumption that the dynamical symmetry is generated by a GKT. This procedure will give an extension of the related integral theorem.

Theorem 1. Suppose that $H_{b_1\cdots b_m}$ is a KT and that $K_{a_1\cdots a_p}$ is a GKT. Then the quantity

$$\begin{aligned}&\left(m\nabla_{b_1}K^i{}_{a_2\cdots a_p}H_{ib_2\cdots b_m} + K^i{}_{a_2\cdots a_p}\nabla_i H_{b_1\cdots b_m}\right)\dot{q}^{a_2}\cdots\dot{q}^{a_p}\dot{q}^{b_1}\cdots\dot{q}^{b_m}\\ &\quad - m\Gamma(\tau)H_{b_1\cdots b_m}\dot{q}^{b_1}\cdots\dot{q}^{b_m}\end{aligned} \tag{36}$$

is conserved.

Proof. Consider equation (31), where ϕ is substituted by $H_{b_1\cdots b_m}\dot{q}^{b_1}\cdots\dot{q}^{b_m}$. Assume that K^a and η^a are given by equations (24) and (25), respectively. In view of the identity

$$\begin{aligned}\nabla_{b_1}K^a{}_{a_2\cdots a_p}\dot{q}^{b_1}\dot{q}^{a_2}\cdots\dot{q}^{a_p} = \Big[\Big(\partial K^a{}_{a_2\cdots a_p}/\partial q^{b_1}\Big) &\\ - (p-1)\Gamma_{b_1a_2}{}^c K^a{}_{ca_3\cdots a_p} &\\ + \Gamma_{b_1c}{}^a K^c{}_{a_2\cdots a_p}\Big]\dot{q}^{b_1}\dot{q}^{a_2}\cdots\dot{q}^{a_p}&\end{aligned}$$

we deduce that the conserved quantity (31) may be written as

$$Y\left(H_{b_1\cdots b_m}\dot{q}^{b_1}\cdots\dot{q}^{b_m}\right)=\left[K^a{}_{a_2\cdots a_p}\left(\partial H_{b_1\cdots b_m}/\partial q^a\right)\right.$$

$$\left.+\left(\nabla_{b_1}K^i{}_{a_2\cdots a_p}-\Gamma_{b_1a}{}^iK^a{}_{a_2\cdots a_p}\right)mH_{ib_2\cdots b_m}\right]$$

$$\times\dot{q}^{a_2}\cdots\dot{q}^{a_p}\dot{q}^{b_1}\cdots\dot{q}^{b_m}-m\Gamma(\tau)H_{b_1\cdots b_m}\dot{q}^{b_1}\cdots\dot{q}^{b_m}$$

Comparing with the definition of $\nabla_a H_{b_1\cdots b_m}$ we conclude that the quantity (36) yields a first integral of motion. ∎

To comment on the meaning of the theorem we make the following remarks. Firstly, the conserved quantity (36) depends on the scalar $\Gamma(\tau)$ which can be evaluated by integration of equation (27). Therefore, the computation of the first integral (36) is not straightforward, since it involves the search for the solution of a partial differential equation.

Secondly, if the given KT coincides with the metric tensor g_{ab} then the first integral (36) assumes the form

$$2\nabla_{b_1}K_{b_2a_2\cdots a_p}\dot{q}^{b_1}\dot{q}^{b_2}\dot{q}^{a_2}\cdots\dot{q}^{a_p}-2\Gamma(\tau)g_{ab}\dot{q}^a\dot{q}^b \tag{37}$$

In particular, when we consider null geodesics for which $g_{ab}\dot{q}^a\dot{q}^b=0$ identically, the conserved quantity (37) reduces to a polynomial of order $p+1$, showing that $\nabla_{(b_1}K_{b_2a_2\cdots a_p)}$ is a conformal Killing tensor.

Thirdly, let us suppose that the tensor field $K_{a_1\cdots a_p}$ is a KT. Comparing equations (26) and (27) with the definition (29) of KT, it follows that $\Gamma(\tau)$ can be set equal to zero. Moreover, equation (29) yields the identity

$$\left(\nabla_i K_{b_1a_2\cdots a_p}+p\nabla_{b_1}K_{ia_2\cdots a_p}\right)\dot{q}^{b_1}\dot{q}^{a_2}\cdots\dot{q}^{a_p}=0 \tag{38}$$

In view of equation (38) the conserved quantity (36) may be written as

$$(1/p)\left(pK^i{}_{a_2\cdots a_p}\nabla_i H_{b_1\cdots b_m}-mH^i{}_{b_2\cdots b_m}\nabla_i K_{b_1a_2\cdots a_p}\right)$$

$$\times\dot{q}^{a_2}\cdots\dot{q}^{a_p}\dot{q}^{b_1}\cdots\dot{q}^{b_m} \tag{39}$$

showing that the first integral (36) is generated by the Schouten–Nijenhuis bracket of the given Killing tensors, up to a constant factor (Sommers, 1973). Using the terminology of analytical mechanics we may say that the conserved quantity (39) is equivalent to the Poisson bracket of the first integrals $K_{a_1\cdots a_p}\dot{q}^{a_1}\cdots\dot{q}^{a_p}$ and $H_{b_1\cdots b_m}\dot{q}^{b_1}\cdots\dot{q}^{b_m}$ (Katzin et al., 1968).

6. DYNAMICAL SYMMETRIES AND NOETHER-TYPE CONSERVED QUANTITIES

In this section we examine a different class of first integrals which are also interesting insofar as they give a further contribution to the analysis of the relations between conserved quantities and dynamical symmetries. Our approach is based on a recent result by Lutzky (1979a) stating that a vector field, and in particular a dynamical symmetry Y, gives rise to a Noether-type conserved quantity of the form

$$\psi = (\tau\dot{q}^a - K^a)(\partial L/\partial\dot{q}^a) - \tau L + f \tag{40}$$

if and only if the function f satisfies the equation

$$Y(L) + \Gamma(\tau)L = \Gamma(f) \tag{41}$$

Lutzky argues that it is not useful to allow f to depend on $\dot{q}$, for, if it did, we could always associate any conserved quantity with any arbitrary transformation. However, our conditions are not so general as in Lutzky's work. For instance, the field Y is a dynamical symmetry constructed by a well-defined procedure; moreover, we only aim at writing down the explicit expression of at least one first integral associated with Y. Accordingly, we allow f to depend on s, q, and $\dot{q}$, showing that under these conditions equation (41) can be solved for f. Namely, we have the following theorem.

Theorem 2. (a) To every PC K^a corresponds a Noether-type conserved quantity of the form

$$\psi = \tau g_{ab}\dot{q}^a\dot{q}^b - K_a\dot{q}^a + \frac{1}{2}s\left(\nabla_a K_b + \nabla_b K_a - \frac{4\nabla_c K^c}{n+1}g_{ab}\right)\dot{q}^a\dot{q}^b \tag{42}$$

(b) To every GKT $K_{a_1\cdots a_p}$ corresponds a Noether-type conserved quantity of the form

$$\psi = \tau g_{ab}\dot{q}^a\dot{q}^b - K_{a_1\cdots a_p}\dot{q}^{a_1}\cdots\dot{q}^{a_p} + s\Big[\nabla_a K_{a_1\cdots a_p}\dot{q}^a\dot{q}^{a_1}\cdots\dot{q}^{a_p} - \Gamma(\tau)g_{ab}\dot{q}^a\dot{q}^b\Big] \tag{43}$$

Proof. (a) Consider the dynamical symmetry (5) associated with the PC K^a, such that $\Gamma(\tau) = 2\nabla_c K^c/(n+1)$ and η^a is given by equation (13). In order to determine f from equation (41) we recall that $Y(L)$ is the conserved quantity (35). It is also to be remarked that $\Gamma(L)$ vanishes identically, so that $\Gamma(\tau)L = \Gamma(\tau L)$. It follows from the above observations that f may be written as

$$f = \frac{1}{2}\tau g_{ab}\dot{q}^a\dot{q}^b + \frac{1}{2}s\left(\nabla_a K_b + \nabla_b K_a - \frac{4\nabla_c K^c}{n+1}g_{ab}\right)\dot{q}^a\dot{q}^b \qquad (44)$$

Substituting equation (44) into equation (40) and recalling the expressions of L and η^a, we conclude that the conserved quantity (40) assumes the form (42).

(b) The proof is omitted since it is very similar to the proof of part (a). ■

Referring to the conserved quantities (42) and (43), we do want to point out that the existence of these first integrals cannot be proved in a purely relativistic context, where, at most, one can find conserved quantities concomitant with a GKT under very restrictive ad hoc conditions (Caviglia et al., 1982a). The point is that one needs the concept of extended tangent space in order to introduce the function τ which appears both in (42) and in (43). Actually, assuming that the function τ has been determined, we may recall the discussion of the previous section to conclude that every GKT always gives rise to at least two first integrals of motion.

An inspection of equations (42) and (43) reveals that they depend explicitly on s, so that the knowledge of a further family of $2n-1$ functionally independent first integrals of the form $f(q, \dot{q}) = \text{const}$ is sufficient to determine the geodesics of the given metric.

In general, the dynamical symmetry (23), say, $Y = \partial/\partial s$, when acting on a first integral of motion gives rise to another first integral. By applying this already known result (Katzin et al., 1977) it may be proved that the conserved quantities (42) and (43) identify the first integrals (35) and (37), respectively, thus showing the consistency of our analysis.

Suppose now that K^a is a Killing vector and that $K_{a_1\cdots a_p}$ is a KT. Let us choose $\tau = 0$, consistently with equations (18) and (27). Then the quantities (42) and (43) reduce to the well-known conserved quantities naturally associated with Killing vectors and KTs. No further information is gained if one allows $\tau = \text{const}$ in the case of the Killing vector or $\tau = \alpha s + \beta$ in the case of the KT, as can be easily seen by substitution into equations (42) and (43), respectively.

7. DISCUSSION

In this work we have described a procedure aiming at the generation of dynamical symmetries of the Euler–Lagrange equations deduced from a regular Lagrangian $L = \frac{1}{2}g_{ab}\dot{q}^a\dot{q}^b$. We have considered only dynamical symmetries such that their natural projection onto the configuration manifold M yields a vector field K^a of M or is generated by a totally symmetric tensor field $K_{a_1\cdots a_p}$, in the sense that it may be written as $K^a{}_{a_2\cdots a_p}\dot{q}^{a_2}\cdots\dot{q}^{a_p}$. The fundamental motivation for this choice has come from recently established results in the field of general relativity, where it has been shown that geometric objects of the forms described above may play the role of generators of first integrals of geodesic motion. Then we have proved that the class of tensor fields associated with dynamical symmetries coincides with the family of generalized Killing tensors. A geometric interpretation of generalized Killing tensors as generators of infinitesimal variations of arbitrary geodesics has been given elsewhere (Caviglia et al., 1982a, b).

Subsequently we have explored the possibility of generating new first integrals of motion through the deformation of a given one via a dynamical symmetry. This technique has led to a proper extension of a class of conserved quantities that were already known in general relativity. However, the most striking result consists in the determination of Noether-type conserved quantities whose explicit form can be specified from a knowledge of the dynamical symmetry, without any further integration being required.

It seems that, besides clarifying a few technical problems concerning the relations between dynamical symmetries and Noether-type conserved quantities, our approach shows the advantage of looking at the methods worked out in general relativity from a more abstract viewpoint, in order to obtain the maximum possible amount of information. Conversely, it also suggests efficient procedures for the determination of dynamical symmetries and of the related conserved quantities in analytical mechanics.

In this work we have not examined the possibility of computing constants of motion by using the formula for the Poisson brackets in Lagrangian mechanics (Sarlet et al., 1981), since the results obtained by this method are straightforward from a theoretical viewpoint, even though they may turn out to be useful in practical applications. Similarly, we have not analyzed the possibility of generating first integrals by repeated application of the deformation technique described in Section 5. To give a little feeling of possible results, let us consider two PCs, say, K^a and H^a, and denote by Y and Z, respectively, the associated dynamical symmetries. Then it follows that the quantity $ZY(L)$ is conserved and is given by

$$ZY(L) = \left[\frac{1}{2}\mathfrak{L}_H\left(\nabla_a K_b + \nabla_b K_a - \frac{4\nabla_c K^c}{n+1} g_{ab}\right)\right.$$
$$\left. -2\nabla_c H^c\left(\nabla_a K_b + \nabla_b K_a - \frac{4\nabla_c K^c}{n+1} g_{ab}\right)\right]\dot{q}^a\dot{q}^b$$

In particular, the tensor field enclosed in square brackets is a second-order Killing tensor.

Finally, a comment concerning the lack of specific examples is in order. As a matter of fact, it should be observed that in the existing literature on the connections between symmetry properties and conserved quantities one can easily find several applications that may be regarded as special cases illustrating the procedures described in this paper. They have not been reported here, owing to the fact that they can be easily found in the "relativistic" papers referred to at the end of this section. More involved applications concerning Noether-type conserved quantities are still under investigation.

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